

## Supplement to Computing an Arithmetic Constant Related to the Ring of Gaussian Integers

By F. Gramain and M. Weber

Appendix 1 : BASIC program for  $\delta_{1000}$

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10 NP = 0 ; YC = 0 ; NX = 0 ; R2 = 0 ; DN = 0 ; C = 0 ; F2 = 0 ; W2 = 0 :
    EP = .000 001 : K = 0 : X = 0 : Y = 0 : W1 = 0
20 S = 0 : G = 0 : IR = 0 : L2 = 0 : L = 0 : U = 1 : B2 = 0 : B = 0 : RM = 0 :
    M1 = 0 : HI = 0 : M2 = 0 : LX = 0 : V2 = 0
30 XB = 0 : C2 = 0 : D = 2 : YX = 0 : N = 1000 : YB = 0 : BX = 0 : L1 = 0 :
    Z = 0 : P = 1.0001
40 DIM RR(N) : RR(D) = 4 : L = D*SQR(N/W) : L2 = L*L : L1 = L2/4
100 FOR XB = Z TO L/SQR(D) : B2 = XB*XB : LX = XB : IF XB = Z THEN LX = U
110 FOR YB = LX TO SQR(L2-B2) : B = B2+YB*YB : PRINT TT/8 ; XB, YB
120 FOR XC = Z TO L : BX = BX*XC : C2 = XC*XC : YX = YB*XC : M1 = Z :
    M2 = (B-C2)/P : IF M2 > Z THEN M1 = SQR(M2)
130 RM = Z : HI = P*(B-XB*XC)/YB : IF HI < Z THEN NEXT YB : NEXT : GO TO 260
140 RM = SQR(L2-C2) : IF RM > HI THEN RM = HI
150 FOR YC = INT(RM) TO M1 STEP -U : C = C2+YC*YC
160 DN = D*(YX-XB*YC) : IF DN = Z THEN NEXT : NEXT : NEXT : GO TO 260
170 NX = YB*C-YC*B : NY = BX-XB*C : S = NX*DN+NY*NY : G = DN*DN : R2 = S/G
180 IF R2 > L1 THEN NEXT : NEXT : NEXT : NEXT : GO TO 260
190 Y = NY/DN : X = NX/DN : NP = Z : F1 = SQR(R2+EP) : W1 = W-F1
200 IF W1 < INT(W1) THEN W1 = INT(W1+U)
210 FOR K = W1 TO X+F1 : F2 = X-K : F2 = SQR(R2+EP-F2*F2) : W2 = Y-F2
220 NP = NP+INT(Y+F2)-INT(W2) : NEXT
230 IF NP > N THEN NP = N
240 IR = G/S : IF RR(NP) > IR THEN NEXT : NEXT : NEXT : GO TO 260
250 RR(NP) = IR : NEXT : NEXT : NEXT
260 OPEN 1 : FOR K = N TO U STEP -U : PRINT = 1, RR(K) : NEXT : CLOSE 1 : END
  
```

This first program computes the  $r_k^{-2}$  and stores their values in the table  $RR(K)$ , then in the file # 1. The latter is exploited by the program below which computes the  $r_k''^{-2}$  (the conjecture in §.1 is that  $r_k'' = r_k$  for  $k \geq 3$ ) and then computes  $\delta_n$  for  $n = SM$ .

```

10 K = 0 : N = 0 : PT = 0 : TT = 0 : L = 0 : MX = 0 : SM = 0 : E8 = ""
20 INPUT "TAB LENGTH" ; L : DIM T(L) : OPEN 1 : FOR K = L TO 2 STEP -1 :
    INPUT # 1, T(K) : NEXT : CLOSE 1
30 PT = 0 : K = L
40 IF T(K) = 0 THEN K = K-1 : GO TO 40
50 MX = K : PRINT "MAX.TAB = " ; MX : FOR K = MX TO 2 STEP -1 : IF T(K) < PT
    THEN T(K) = PT : NEXT : GO TO 70
60 PT = T(K) : NEXT
70 INPUT "TOTAL FROM 2 TO" ; SM : IF SM > MX THEN 70
80 TT = 0 : FOR K = SM TO 2 STEP -1 : TT = TT+T(K) : NEXT
90 PRINT SM ; TT/TT ; TT/TT -LOG(SM) : STOP : GO TO 50
  
```

Appendix 2 : Prospecting exceptional disks.

The following table gives for integers of the form  $2r = 2m+1$  ( $1 \leq m \leq 21$ ) having only prime factors congruent to 3 (mod 4) the number  $k$  of integer points belonging to the closed disk of center  $1/2$  and radius  $r$ ; it shows when  $r \leq \sqrt{(k-1)/\pi}$  and gives  $r_k^0$  in this case. One checks that  $r > r_k^0$ , therefore, for  $k \neq 1500$ , there exists no exceptional disk.

$2r$	$k$	$r \leq \sqrt{(k-1)/\pi}$	$r_k^0$
3	8	NO	
7	40	YES	3.424 13...
$9=3^2$	62	NO	
11	96	NO	
19	280	NO	
$21=3 \cdot 7$	346	NO	
23	412	NO	
$27=3^3$	576	YES	13.45362...
31	756	YES	15.44445...
$33=3 \cdot 11$	850	NO	
43	1448	NO	

$= r_{40}^0$   
 $= r_{577}^0$   
 $= r_{756}^0$

Appendix 3 : Table of  $r_k$  ( $2 \leq k \leq 300$ )

For each value of  $k$  ( $2 \leq k \leq 300$ ) the coordinates of the center of some minimal circle  $\Gamma_k$  passing through the origin and the integer points  $b_k$  and  $c_k$  are  $x = N(x)/D$  and  $y = N(y)/D$ , and its squared radius is  $r_k^2 = N(r^2)/D^2$ . The last column gives an approximation to  $r_k$ .

If the line of index  $k_0$  is empty there exists a smallest  $k_1 > k_0$  such that the line of index  $k_1$  is nonempty and then the minimal disks of indices  $k_0, k_0+1, \dots, k_1$  are identical. For instance,  $r_{18} = r_{20} = r_{21} = \sqrt{5}$ . Notice that not all the centers of the minimal disks are on the symmetry axes of the lattice  $\mathbb{Z}[i]$  (see for instance  $D_{17}$ ).

$k$	$b$	$c$	$N(x)$	$N(y)$	$D$	$N(r^2)$	$r_k$	
3	1		0	1	2	1	0.5	
4	1	1	1	1	2	2	0.7071 06781	
5	-1+	1+	0	1	1	1	1	
6	2+	1	1	2	2	5	1.1180 33989	
7	-1+	1+	0	5	4	25	1.25	
8	2+	2	1	1	1	2	1.4142 13562	
10								
11	-2+	1+	-1	3	2	10	1.5811 38830	
12	-2+	1+	-7	17	10	338	1.8384 7651	
13	-2+	1+	-3	11	6	130	1.9802 52371	
14	-2+	1+	1	5	4	5	2.0615 52813	
15	4+	1	1	4	2	17	2.0615 52813	
16	-2+	3+	0	13	6	169	2.1666 66667	
17								
18	-3+	1+	-1	2	1	5	2.2360 67977	
19	-4+	1+	-3	4	2	25	2.5	
20	-3+	2+	-1	5	2	26	2.5495 093757	
21	5+	3+	2	5	2	29	2.6925 82404	
22	-1+	5+	21	103	38	11	050 2.7662 88953	
23	-3+	4+	-2	11	4	125	2.7950 84972	
24								
25	-4+	1+	-3	5	2	34	2.9154 79947	
26	-4+	2+	-11	29	10	962	3.1016 12484	
27								
28	-4+	2+	-1	3	1	10	3.1622 77650	
29	6+	3	3	16	2	48	3.1752 01965	
30	-5+	1+	-11	17	6	410	3.3717 43785	
31	-1+	6+	61	159	58	39	442 3.4241 35667	
32	-4+	5+	-10	33	10	1	189 3.4481 87930	
33								
34	-6+	1+	-5	5	2	50	3.5355 33906	
35	-5+	1+	-2	3	1	13	3.6055 51275	
36	-4+	2+	-7	17	5	338	3.6769 55262	
37	-2+	5+	89	227	66	59	450 3.6942 98588	
38	-4+	3+	-5	37	10	1	354 3.7336 30941	
39								
40	-6+	1+	-3	5	2	58	3.8078 86553	
41	-5+	2+	-3	7	2	58	3.8078 86553	
42								
43	-6+	3+	-4	7	2	65	4.0311 28874	
44	-5+	1+	-1	4	1	17	4.1231 06626	
45	-4+	2+	-1	4	1	1	105 4.1591 25335	
46	-2+	1+	-4	33	8	3	445 4.1924 37311	
47	-4+	1+	-14	57	14	5	50	
48								
49								
50								
51								
52								
53								
54								
55								
56								
57								
58								
59								
60								

k	b	c	N(x)	N(y)	D	N(r <sup>2</sup> )	Γ <sub>k</sub>
51	6+	6	3	3	1	18	4, 2426 40687
52	-6+ 5+	2+ 2+	-41	85	22	8 906	4, 2896 18490
53	-7+ 5+	1+ 4+	-211	193	66	81 770	4, 3326 44573
54	-5+ 7+	2+ 6+	-41	87	22	9 250	4, 3716 78196
55	-7+ 5+	1+ 5+	-15	16	5	481	4, 3863 42440
56							
57							
58							
59	-6+ 6+	2+ 2+	-2	4	1	20	4, 4721 35955
60	-8+ 4+	1+ 2+	-7	6	2	85	4, 6097 72229
61	-3+ 6+	6+ 5+	47	151	34	4 6513	4, 6513 38313
62	-6+ 7+	1+ 7+	-35	55	14	4 250	4, 6565 73147
63	-4+ 8+	4+ 4+	-2	14	3	200	4, 7140 45208
64	-8+	5	5	8	2	89	4, 7169 90566
65							
66	-6+ 6+	3+ 3+	-3	9	2	90	4, 7434 15490
67	-5+ 8+	3+ 3+	-41	119	26	15 862	4, 8489 51809
68	-7+ 6+	2+ 6+	-13	23	6	854	4, 8991 24579
69	-5+ 7+	6+ 7+	162	423	85	179 850	4, 9257 68368
70	-2+ 9+	1+ 6+	162	269	62	93 925	4, 9430 94491
71	-6+ 7+	3+ 7+	-67	51	14	4 810	4, 9538 68824
72	-6+ 7+	3+ 7+	-21	67	14	4 930	5, 0152 82768
73	-1+ 10+	1+ 10+	0	101	20	10 201	5, 0520 65544
74	-5+ 8+	4+ 6+	-59	308	62	98 345	5, 0580 65544
75	-8+ 3+	1+ 1	-67	89	22	12 410	5, 0636 44524
76							
77							
78							
79	-5+ 7+	6+ 7+	162	423	85	179 850	4, 9257 68368
80	-2+ 9+	1+ 6+	162	269	62	93 925	4, 9430 94491
81	-6+ 7+	3+ 7+	-67	51	14	4 810	4, 9538 68824
82	-1+ 10+	1+ 10+	0	101	20	10 201	5, 0152 82768
83	-5+ 8+	4+ 6+	-59	308	62	98 345	5, 0580 65544
84	-8+ 3+	1+ 1	-67	89	22	12 410	5, 0636 44524
85							
86							
87							
88	-6+ 6+	4+ 4+	-1	5	1	26	5, 0990 19514
89	-8+ 6+	2+ 3+	-111	152	36	35 425	5, 2282 02840
90	-7+ 6+	3+ 3+	-49	127	26	18 530	5, 2355 74715
91	-3+ 7+	7+ 6+	247	661	134	497 930	5, 2659 81732
92	-7+ 7+	3+ 6+	-13	29	6	1 010	5, 2967 49528
93	-5+ 9+	4+ 4+	-17	73	14	5 618	5, 3538 08486
94							
95							
96	-7+ 7+	3+ 3+	-2	5	1	29	5, 3851 64807
97	-8+ 5+	3+ 5+	-25	49	10	3 028	5, 3899 69616
98	-8+ 7+	2+ 7+	-42	69	14	5 993	5, 5276 08251
99	-10+ 4+	1+ 4+	-18	13	4	433	5, 5589 08628
100							
101							
102							
103							
104							
105	-8+ 6+	3+ 4+	-5	10	2	125	5, 5901 69944
106							
107							
108							
109							
110							
111	-10+ 5+	1+ 2+	-9	7	2	130	5, 7008 77125
112	-10+ 6+	1+ 2+	-121	93	26	23 290	5, 8636 39062
113	-9+ 7+	1+ 1	-29	37	8	2 210	5, 8763 29637
114	-3+ 8+	8+ 7+	383	923	170	1 006 378	5, 9010 81947
115	-7+ 8+	4+ 8+	-6	23	4	565	5, 9424 32162
116	-8+ 9+	1+ 1	-127	161	34	42 050	6, 0312 04897
117							
118							
119							
120	-8+ 8+	3+ 3+	-5	11	2	146	6, 0415 22987

k	b	c	N(x)	N(y)	D	N(r <sup>2</sup> )	Γ <sub>k</sub>
121	-7+ 7+	5+ 5+	-1	6	1	37	6, 0627 62530
122	-11+ 3+	1+ 2+	-49	37	10	3 770	6, 1400 32573
123	-8+ 9+	3+ 4+	-355	635	118	529 250	6, 1652 17853
124	-8+ 9+	2+ 9+	-54	97	18	12 325	6, 1676 67589
125	-9+ 7+	3+ 4+	-115	205	38	55 250	6, 1856 10145
126	-9+ 7+	3+ 7+	-21	188	7	1 885	6, 2023 69527
127	-8+ 8+	4+ 7+	-47	129	22	18 850	6, 2406 95559
128	-11+	6	6	11	2	157	6, 2649 82043
129	-6+ 10+	4+ 10+	-5	31	5	986	6, 2801 27388
130	-11+ 6+	1+ 2+	-71	53	14	7 650	6, 3285 87932
131	-11+ 6+	1+ 6+	-60	47	12	5 809	6, 3513 35814
132	-2+ 11+	7+ 10+	389	1173	194	1 527 230	6, 3702 63748
133	-7+ 10+	3+ 10+	-40	121	20	16 241	6, 3720 09108
134							
135							
136							
137	-9+ 9+	1+ 1	-4	5	1	41	6, 4031 24237
138							
139	-12+ 4+	1+ 3+	-11	7	2	170	6, 5192 02405
140	-12+ 5+	1+ 3+	-457	289	82	292 370	6, 5940 54791
141	-12+ 5+	1+ 13+	-13	171	26	29 410	6, 5959 01542
142	-2+ 13+	1+ 13+	-151	684	106	490 637	6, 6081 99325
143	-10+ 4+	2+ 2+	-25	39	7	2 146	6, 6178 48470
144							
145							
146							
147	-8+ 8+	5+ 5+	-3	13	2	178	6, 6708 32030
148							
149	-12+ 6+	1+ 2+	-11	8	2	185	6, 8007 35255
150	-4+ 13+	4+ 11+	-127	644	96	430 865	6, 8375 32056
151	-9+ 9+	4+ 5+	-49	113	18	15 170	6, 8425 86328
152	-6+ 11+	6+ 9+	-21	274	40	75 517	6, 8700 89155
153	-11+ 7+	1+ 1	-13	16	3	425	6, 8718 42710
154	-3+ 12+	7+ 11+	-33	119	27	291 890	6, 9265 18613
155	-7+ 11+	4+ 11+	-33	119	27	291 890	6, 9265 18613
156	-7+ 12+	4+ 9+	-197	333	72	135 246	6, 9933 06758
157	-12+ 7+	7+ 10+	157	1631	242	2 890 898	7, 0058 84698
158	-10+ 7+	4+ 7+	-152	89	11	9	7, 0294 56971
159	-12+ 6+	2+ 4+	-5	19	5	1 261	7, 0710 67812
160	-13+ 5+	1+ 5+	-30	19	5	1 261	7, 1021 12362
161							
162							
163	-1+ 10+	1+ 1	-9	11	2	202	7, 1063 35200
164							
165	-10+ 8+	4+ 5+	-6	13	2	205	7, 1589 10530
166	-3+ 14+	1+ 14+	-28	199	28	40 385	7, 1771 49825
167	-5+ 11+	8+ 9+	281	1893	266	3 862 410	7, 1945 28327
168	-10+ 10+	2+ 2+	-4	6	1	52	7, 2111 02551
169	-6+ 13+	3+ 12+	-157	511	74	285 770	7, 2239 81043
170	-10+ 10+	3+ 4+	-85	85	14	10 250	7, 2315 91693
171	-8+ 11+	5+ 9+	-499	1773	254	3 392 530	7, 2515 04685
172	-11+ 4+	2+ 2+	-121	181	30	47 402	7, 2573 33457
173							
174							
175							
176							
177	-9+ 9+	5+ 5+	-2	7	1	53	7, 2601 09809
178	-5+ 14+	3+ 9+	-101	327	46	117 135	7, 2600 56699
179	-10+ 9+	10+ 9+	-363	561	38	80 092	7, 4433 23316
180	-4+ 10+	10+ 9+	363	561	136	1 034 853	7, 4770 87985

k	b	c	N(x)	N(y)	D	N(x <sup>2</sup> )	r <sub>k</sub>
181	-1+9	1+8	-81	67	14	11 050	7.5064 98585
182	-1+14	7+13	491	1597	222	2 791 410	7.5030 14725
183	-12+8	1+1	-24	29	5	1 417	7.5236 13090
184	-15	2+15	2	15	2	1 229	7.5653 72975
185	-14+5	1+5	-65	39	10	5 746	7.5892 37453
186	-13	8	13	13	2	233	7.5911 37453
187	-13	8	13	13	2	233	7.5911 37453
188	-8+13	4+5	-158	315	46	124 189	7.5911 37453
189	-2+11	11+10	1181	1817	282	4 696 250	7.5911 37453
190	-4+12	9+11	83	281	38	85 650	7.5911 37453
191	-14+6	1+6	-39	25	6	252	7.5911 37453
192	-4+14	7+12	79	1128	146	1 278 655	7.5911 37453
193	-7+12	6+12	-2	31	4	965	7.5911 37453
194	-11+11	5+8	-27	28	5	1 517	7.5911 37453
195	-13+8	2+9	-1207	1155	214	2 790 874	7.5911 37453
196	-10+10	5+9	-37	103	14	11 978	7.5911 37453
197	-5+12	5+9	-251	693	94	543 250	7.5911 37453
198	-2+14	8+13	331	1033	138	1 176 690	7.5911 37453
199	-13+8	2+8	-44	45	8	3 561	7.5911 37453
200	-13+8	2+8	-44	45	8	3 561	7.5911 37453
201	-13+8	2+8	-44	45	8	3 561	7.5911 37453
202	-13+8	2+8	-44	45	8	3 561	7.5911 37453
203	-13+8	2+8	-44	45	8	3 561	7.5911 37453
204	-13+8	2+8	-44	45	8	3 561	7.5911 37453
205	-13+8	2+8	-44	45	8	3 561	7.5911 37453
206	-13+8	2+8	-44	45	8	3 561	7.5911 37453
207	-13+8	2+8	-44	45	8	3 561	7.5911 37453
208	-13+8	2+8	-44	45	8	3 561	7.5911 37453
209	-13+8	2+8	-44	45	8	3 561	7.5911 37453
210	-13+8	2+8	-44	45	8	3 561	7.5911 37453
211	-13+8	2+8	-44	45	8	3 561	7.5911 37453
212	-13+8	2+8	-44	45	8	3 561	7.5911 37453
213	-13+8	2+8	-44	45	8	3 561	7.5911 37453
214	-13+8	2+8	-44	45	8	3 561	7.5911 37453
215	-13+8	2+8	-44	45	8	3 561	7.5911 37453
216	-13+8	2+8	-44	45	8	3 561	7.5911 37453
217	-13+8	2+8	-44	45	8	3 561	7.5911 37453
218	-13+8	2+8	-44	45	8	3 561	7.5911 37453
219	-13+8	2+8	-44	45	8	3 561	7.5911 37453
220	-13+8	2+8	-44	45	8	3 561	7.5911 37453
221	-13+8	2+8	-44	45	8	3 561	7.5911 37453
222	-13+8	2+8	-44	45	8	3 561	7.5911 37453
223	-13+8	2+8	-44	45	8	3 561	7.5911 37453
224	-13+8	2+8	-44	45	8	3 561	7.5911 37453
225	-13+8	2+8	-44	45	8	3 561	7.5911 37453
226	-13+8	2+8	-44	45	8	3 561	7.5911 37453
227	-13+8	2+8	-44	45	8	3 561	7.5911 37453
228	-13+8	2+8	-44	45	8	3 561	7.5911 37453
229	-13+8	2+8	-44	45	8	3 561	7.5911 37453
230	-13+8	2+8	-44	45	8	3 561	7.5911 37453
231	-13+8	2+8	-44	45	8	3 561	7.5911 37453
232	-13+8	2+8	-44	45	8	3 561	7.5911 37453
233	-13+8	2+8	-44	45	8	3 561	7.5911 37453
234	-13+8	2+8	-44	45	8	3 561	7.5911 37453
235	-13+8	2+8	-44	45	8	3 561	7.5911 37453
236	-13+8	2+8	-44	45	8	3 561	7.5911 37453
237	-13+8	2+8	-44	45	8	3 561	7.5911 37453
238	-13+8	2+8	-44	45	8	3 561	7.5911 37453
239	-13+8	2+8	-44	45	8	3 561	7.5911 37453
240	-13+8	2+8	-44	45	8	3 561	7.5911 37453